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Convective instability of moist gas in a porous medium

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Abstract—Flow and heat transfer in a porous medium filled with an ideal gas of 100% humidity are strongly coupled. The transitions between the conductive and convective regimes can be found by stability analysis of the governing equations. A dimensionless Rayleigh number controls the heat and flow regime. Stability conditions obtained by perturbation analysis show that the critical Rayleigh number depends heavily on the vapor pressure. The moist gas studied here is much less stable than a dry ideal gas, because the latent heat carried by a warm moist gas is much greater than the sensible heat.

INTRODUCTION

THE ONSET of thermal instability in horizontal layers of fluid heated from below is a classical problem and has been studied extensively in both pure fluids and porous media. The stability of the system can be characterized mathematically by the numerical value of a dimensionless parameter called the Rayleigh number [1]. The theory was applied to liquids in porous media by Horton and Rogers [2] and Lapwood [3]. Saatdjian [4] and Nield [5] extended the solution to a porous medium containing an ideal gas.

The classical results predict that if the system's Rayleigh number is less than a critical value, conduction will be the only mechanism for heat transfer. If the Rayleigh number exceeds its critical number, a transition from pure conduction to conduction–convection heat transfer will occur. At yet higher Rayleigh numbers, new flow patterns will occur and eventually regular flow patterns will disappear and the system will enter a chaotic state.

Coupled heat transfer and fluid flow in unsaturated media has been a little known area in the past. Only recently has it begun to draw the attention of some researchers. Because of the nonlinearity of the governing equations, it is a difficult and challenging problem. Plumb [6] discussed the modeling of convection in unsaturated porous media with and without boiling or condensation. The particular problem of drying of porous media has been surveyed by Plumb [6] and Bories [7]. Tien and Vafai [8] and Nield and Bejan [9] provide general reviews of convection in unsaturated porous media.

The purpose of this paper is to examine the onset of convective gas flow in an unsaturated porous medium containing an ideal gas constrained to remain at 100% relative humidity. The humidity constraint is physically realistic; unsaturated soils and rocks almost always contain some liquid water except very near the ground surface, and this water keeps the gas humidity close to 100% [10]. The geometry studied here is an infinite horizontal layer heated from below.

The motivation for this work comes from a study of the heat and gas transfer in the geological formations near a potential nuclear waste repository at Yucca Mountain, Nevada, U.S.A. The potential repository would be located above the water table in partially saturated tuff. Gas fills most of the larger-diameter pores and fractures and can move through the rock [11, 12]. If nuclear waste is buried at Yucca Mountain, it will add a heat source near the bottom of a permeable layer. The interactions between heat and gas flow under these conditions are the subject of much current research [13–15].

Subsurface flow of moist gas also plays a significant role in formation of sulfuric acid in mine wastes. Heat released by the chemical reaction between oxygen and sulfide minerals stimulates convective gas flow, which carries in more oxygen to continue the reaction.

This study examines the onset of convective gas flow in an infinite horizontal layer of porous medium filled with moist gas by employing a perturbation technique. This technique, which is the usual method of solving convective instability problems, involves three steps. First, we solve the governing equations with no fluid flow (static solution). Second, the static solution is perturbed slightly in as general a manner as possible consistent with the boundary conditions. At this step, appropriate dimensionless parameters are identified and the perturbation equations are reformulated as an eigenvalue problem. Third, we solve this well defined eigenvalue problem to describe the evolution of the perturbations with expressions which are exponential in time. The sign of the exponent determines whether the fluctuations will decay or grow, and thus whether the static solution is stable. Furthermore, the magnitude of the exponent gives a time constant for convective redistribution of heat.

NOMENCLATURE

A_0	dimensionless coefficient
B_0	dimensionless coefficient
c	conversion factor, equal to 4.18×10^7 [erg cal^{-1}]
c_1^0	integral coefficient
c_2^0	integral coefficient
c_p^{gas}	specific heat of gas [cal g^{-1} K^{-1}]
c_p^{rock}	specific heat of rock [cal g^{-1} K^{-1}]
D_0	time constant parameter
E_0	dimensionless coefficient
F	function defined by equation (30)
F_0	dimensionless coefficient
f	function defined by equation (41)
g	gravitational acceleration [cm s^{-2}]
H	thickness of layer [cm]
H_v	heat of vaporization of water [cal g^{-1}]
k	intrinsic permeability of rock [cm^2]
K_t	thermal conductivity of rock [cal K^{-1} cm^{-1} s^{-1}]
l	horizontal wave number
m	vertical wave number
n	porosity [dimensionless]
P	gas pressure [g cm^{-1} s^{-2}]
P_a	defined by $P_a = P - P_v$ [g cm^{-1} s^{-2}]
P_v	vapor pressure of water [g cm^{-1} s^{-2}]
$\mathbf{q}$	gas flux [cm^3 s^{-1}]
R	gas constant [g cm^2 s^{-2} mol^{-1} K^{-1}]
Ra	Rayleigh number
S_h	internal heat source [cal cm^{-3} s^{-1}]
t	time [s]
T	absolute temperature [K]
ΔT	temperature difference between lower and top boundaries [K]
W	solution function of vertical flux
w	dimensionless vertical flux
x	horizontal coordinate [cm]
z	vertical coordinate [cm]
$\mathbf{z}$	downward pointing unit vector.

Greek symbols

α	defined by $g\Omega_a H/R\Delta T$
γ	time rate constant
ζ	dimensionless vertical coordinate
Θ	coefficient of temperature solution
θ	dimensionless temperature
λ_1	defined by equation (19)
λ_2	defined by equation (19)
μ	viscosity of gas [g cm^{-1} s^{-1}]
Π	defined by equation (46)
ρ	density of gas [g cm^{-3}]
ρ_{rock}	density of rock [g cm^{-3}]
ϕ	stream function
Ω_a	molar weight of dry air [g mol^{-1}]
Ω_v	molar weight of water [g mol^{-1}].

Subscripts

0	static solution
s	quantity at surface where $z = 0$
j, k	rank or summation indices
1, 2	mode indices.

Superscripts

$'$	fluctuating quantity
	dimensionless quantity
$*$	instability threshold
$\dagger$	second instability threshold
$\hat{}$	trial solution.

GOVERNING EQUATIONS

Governing equations for heat and gas flow in the porous medium studied in this paper are given by Amter et al. [16]. They consist of four equations, a constitutive relation, Darcy's Law, a volume balance, and an energy balance, as follows:

$$\rho = \frac{1}{RT}[P_v\Omega_v + P_a\Omega_a] \quad (1)$$

$$\mathbf{q} = -\frac{k}{\mu}(\nabla P - g\rho\mathbf{z}) \quad (2)$$

$$\nabla\cdot\mathbf{q} - \mathbf{q}\cdot\left[\left(\frac{1}{T}+\frac{1}{P_a}\frac{dP_v}{dT}\right)\nabla T - \frac{1}{P_a}\nabla P\right] = 0 \quad (3)$$

$$K_t\nabla^2 T - c_p^{gas}\rho\mathbf{q}\cdot\nabla T + \frac{1}{c}\left(1+\frac{P_v}{P_a}\right)\mathbf{q}\cdot\nabla P_a + S_h - \frac{H_v\Omega_v}{RT}\mathbf{q}\cdot\left[\left(1+\frac{P_v}{P_a}\right)\frac{dP_v}{dT}\nabla T - \frac{P_v}{P_a}\nabla P\right] = c_p^{rock}\rho_{rock}(1-n)\frac{\partial T}{\partial t} \quad (4)$$

where ρ is the gas density, R is the gas constant, T is the temperature, Ω_v and Ω_a are the molar weights of water and dry air, g is the acceleration of gravity, k is the intrinsic permeability for gas, and $\mathbf{z}$ is a downward-pointing unit vector. The variable P_v is the vapor pressure of water, which depends only on temperature because of the assumption of 100% humidity. By definition, we have $P_a = P - P_v$. In the energy equation, K_t is the thermal conductivity of the porous medium, c is a conversion factor of 4.18×10^{-7} erg cal^{-1}, c_p^{gas} is the specific heat of gas at constant pressure, c_p^{rock} is the specific heat of rock, H_v is the heat of vaporization of water, n is the porosity and S_h is an internal heat source.

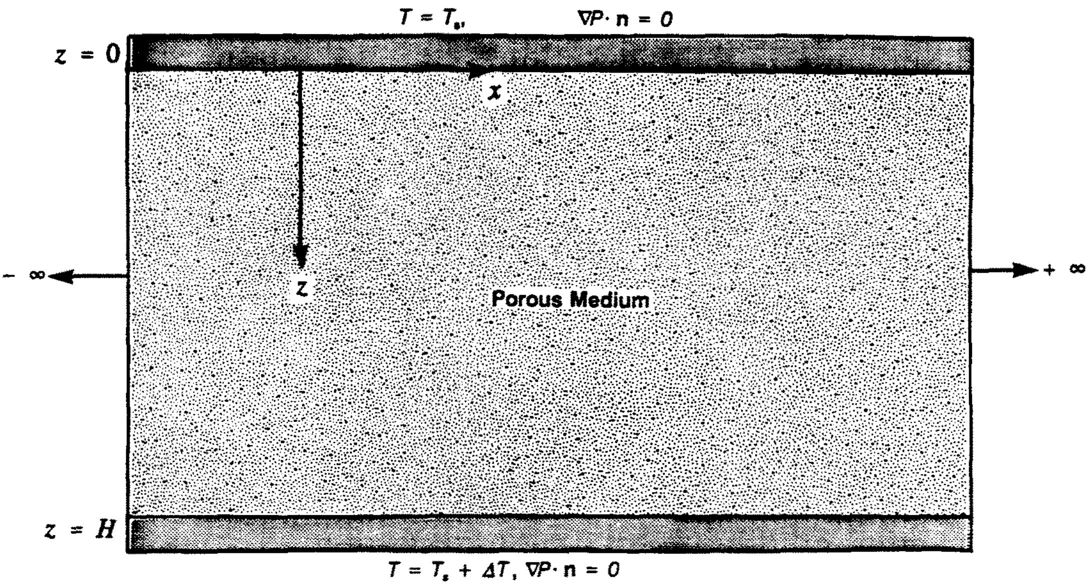


FIG. 1. Illustration of the Horton–Rogers–Lapwood problem: infinite horizontal porous layer heated from below.

For given initial and boundary conditions, equations (1)–(4) can be solved for fields of density ρ, pressure P, temperature T, and gas flux $\mathbf{q}$.

STATIC SOLUTION

We consider a classical problem of the onset of convective gas flow in a porous medium bounded by two horizontal isothermal impermeable planes. This problem is analogous to the Rayleigh–Bénard problem for a viscous fluid and was solved by Horton and Rogers [2] and Lapwood [3] for a porous medium containing a slightly compressible fluid (such as liquid water). Saatdjian [4] and Nield [5] found the solution for a porous medium filled with a non-condensible ideal gas.

The basic equations governing the physical process are equations (1–4). The boundary conditions for the problem with heating from below are illustrated in Fig. 1 and defined as

$$T = T_s, \quad P = P_s, \quad \rho = \rho_s, \quad \mathbf{q} = 0 \quad (z = 0)$$
$$T = T_s + \Delta T, \quad \mathbf{q} = 0 \quad (z = H) \tag{5}$$

where the subscript s refers to values at the upper boundary $z = 0$ (**z** points downward) and H is the thickness of the porous medium.

The static solution with no internal heat source in which the heat transfer is solely by thermal conduction is referred to as the 'conduction state' and is a function of z only. This solution is denoted by the subscript zero. The system is described by the hydrostatic equations

$$\mathbf{q}_0 = 0, \quad T_0 = T_s + \frac{\Delta T}{H} z,$$
$$\rho_0 = \frac{\Omega_a}{RT_0}\left[P_0 - P_{v0}\left(1 - \frac{\Omega_v}{\Omega_a}\right)\right],$$
$$\frac{dP_0}{dz} = \rho_0 g = \frac{g\Omega_a}{RT_0}\left[P_0 - P_{v0}\left(1 - \frac{\Omega_v}{\Omega_a}\right)\right],$$
$$P_{v0} = P_{v0}(T_0). \tag{6}$$

Solving the above equations yields the solution for the distribution of P_0:

$$P_0 = P_s\left(\frac{T_0}{T_s}\right)^{\alpha} - \alpha\left(1 - \frac{\Omega_v}{\Omega_a}\right) T_0^{\alpha} \int_{T_s}^{T_0} \frac{P_v(T)}{T^{\alpha+1}}\, dT \tag{7}$$

with

$$\alpha = \frac{g\Omega_a H}{R\Delta T}. \tag{8}$$

PERTURBATION EQUATIONS

We now examine the stability of the static solution. We expect from the solution of other convective stability problems that the static solution will be unstable if there is a sufficiently large temperature difference across the layer. We consider small two-dimensional disturbances to the static solution because instability occurs first in two dimensions [17]. The perturbation may be written as

$$\mathbf{q} = \mathbf{q}_0 + \mathbf{q}'$$
$$P = P_0 + P'$$

$$\rho = \rho_0 + \rho'$$
$$T = T_0 + T'. \qquad (9)$$

Inserting these forms into the system equations (1)–(4), neglecting all second-order small terms, and subtracting the static solution yields

$$\nabla P' + \frac{\mu}{k}\mathbf{q}' - \rho' g\mathbf{z} = 0 \qquad (10)$$

$$\nabla\cdot\mathbf{q}' - \mathbf{q}'\cdot\left[\left(\frac{1}{T_0} + \frac{1}{P_{a0}}\frac{dP_v}{dT}\bigg|_{T_0}\right)\nabla T_0 - \frac{1}{P_{a0}}\nabla P_0\right] = 0, \qquad (11)$$

$$K_t\nabla^2 T' - \mathbf{q}'\cdot\left\{c_p^{gas}\rho_0\nabla T - \frac{1}{c}\left(1+\frac{P_{v0}}{P_{a0}}\right)\nabla P_{a0} + \frac{H_v\Omega_a}{RT_0}\left[\left(1+\frac{P_{v0}}{P_{a0}}\right)\frac{dP_v}{dT}\bigg|_{T_0}\nabla T_0 - \frac{P_{v0}}{P_{a0}}\nabla P_0\right]\right\} = c_p^{rock}\rho_{rock}(1-n)\frac{\partial T'}{\partial t} \qquad (12)$$

$$\rho' = \frac{\Omega_a}{RT_0}\left[P' - P_v'\left(1-\frac{\Omega_v}{\Omega_a}\right)\right] - \frac{\Omega_a T'}{RT_0^2}\left[P_0 - P_{v0}\left(1-\frac{\Omega_v}{\Omega_a}\right)\right] = \frac{\Omega_a}{RT_0}P' - \frac{\Omega_a T'}{RT_0^2}\left[P_0 + \left(T_0\frac{dP_v}{dT}\bigg|_{T_0} - P_{v0}\right)\left(1-\frac{\Omega_v}{\Omega_a}\right)\right]. \qquad (13)$$

The boundary conditions are

$$\mathbf{q}|_{z=0,H} = 0, \quad T'|_{z=0,H} = 0. \qquad (14)$$

Because all coefficients here are independent of x and t, according to the theory of ordinary linear differential equations with constant coefficients, the solution can be expressed in the form of exponentials in the variables x and t. Hence we have

$$(P', \rho', T') = Re\,[(\hat{P}(z), \hat{\rho}(z), \hat{T}(z))\,e^{ilx+\gamma t}]$$
$$\mathbf{q}' = Re\,[(u(z), w(z))\,e^{ilx+\gamma t}] \qquad (15)$$

where l is the horizontal wavenumber and γ the rate of increase in the size of fluctuation component with wavenumber l. With the above form of solution, the system can be reduced to two equations for the unknown vertical flux and temperature:

$$\left(\frac{d^2}{dz^2} - \frac{A_0+E_0}{H}\frac{d}{dz} - \frac{1}{H}\frac{dA_0}{dz} + \frac{A_0E_0}{H^2} - l^2\right)w = \frac{gP_s\Omega_a l^2 k}{\mu RT_s^2}F_0\hat{T}, \qquad (16)$$

$$\left(K_t\frac{d^2}{dz^2} - D_0 - l^2K_t\right)\hat{T} = \frac{\rho_s c_p^{gas}\Delta T}{H}B_0 w \qquad (17)$$

with dimensionless z-dependent coefficients denoted as

$$A_0 = \frac{\Delta T}{T_0}\left(1+\frac{T_0}{P_{a0}}\frac{dP_v}{dT}\bigg|_{T_0}\right) - \frac{\rho_0 gH}{P_{a0}} \qquad (18)$$

$$B_0 = \frac{\rho_0}{\rho_s} - \frac{\lambda_1 H}{P_s}\left(1+\frac{P_{v0}}{P_{a0}}\right)\frac{dP_{a0}}{dz} + \frac{\lambda_2 H\Delta T}{P_sT_0}\left[\left(1+\frac{P_{v0}}{P_{a0}}\right)\frac{dP_{v0}}{dz} - \frac{P_{v0}}{P_{a0}}\frac{dP_0}{dz}\right],$$

$$\lambda_1 = \frac{P_s}{cc_p^{gas}\rho_s\Delta T}, \quad \lambda_2 = \frac{H_v\Omega_a P_s}{c_p^{gas}\rho_s R(\Delta T)^2} \qquad (19)$$

$$E_0 = \alpha\frac{\Delta T}{T_0} \qquad (20)$$

$$F_0 = \frac{P_0T_s^2}{P_sT_0^2} + \left(\frac{T_0}{P_{v0}}\frac{dP_v}{dT}\bigg|_{T_0} - 1\right)\left(1-\frac{\Omega_v}{\Omega_a}\right)\frac{P_{v0}T_s^2}{P_sT_0^2} \qquad (21)$$

and parameter

$$D_0 = \gamma c_p^{rock}\rho_{rock}(1-n). \qquad (22)$$

Let us discuss the physical effects of these coefficients. The quantity A_0 reflects the effect of the gas compressibility. At low temperature (near the top surface), its value is near zero (less compressible). The parameter increases monotonically to a value on the order of one (more compressible) as the z value tends to the bottom boundary.

The quantity B_0 represents the buoyancy force driven by sensible heat convection, latent heat convection, and change of gas volume. It is unity when there is no vapor pressure, but it tends to infinity near the boiling point.

The quantity E_0 represents the gas density change due to the pressure fluctuation. For parameter values encountered on Earth it is very small.

Finally, the quantity F_0 reflects the enhancement of temperature-caused density change due to the presence of vapor. For the temperatures considered in this study, it ranges from near 1 (at the upper boundary of the system) to teens (near the boiling point).

It is convenient to nondimensionalize equations (16) and (17) by introducing

$$\bar{l} = lH, \quad \zeta = \frac{z}{H}, \quad \bar{w} = \frac{\rho_s Hc_p^{gas}}{K_t}w,$$
$$\theta = \frac{\hat{T}}{\Delta T}, \quad \bar{D} = \frac{H^2D_0}{K_t}. \qquad (23)$$

Equations (16) and (17) then become

$$\left(\frac{d^2}{d\zeta^2} - \bar{D} - \bar{l}^2\right)\theta = B_0\bar{w}, \qquad (24)$$

$$\left(\frac{d^2}{d\zeta^2} - (A_0+E_0)\frac{d}{d\zeta} - \frac{dA_0}{d\zeta} + A_0E_0 - \bar{l}^2\right)\bar{w} = Ra\bar{l}^2F_0\theta \qquad (25)$$

where the Rayleigh number is defined as

$$Ra = \frac{gH\rho_s P_s \Omega_a c_p^{gas} k \Delta T}{\mu K_t R T_s^2}. \quad (26)$$

FIRST APPROXIMATION TO SOLUTION

Equations (24) and (25) are second-order ordinary differential equations, and A_0, B_0, E_0, and F_0 are functions of z. As a first approximation, we regard the coefficients A_0, B_0, E_0 and F_0 as constants. The solutions take the form of an exponential of the dimensionless depth ζ, $\exp(i\bar{m}\zeta)$. Here the vertical dimensionless wavenumber is denoted as $\bar{m}$. Furthermore, for $H = 6.0 \times 10^4$ cm, E_0 is less than $0.082 \ll 1.00$ and can be neglected compared with the dimensionless wavenumber. With these assumptions, from equations (24) and (25) we have

$$(-\bar{m}^2 - \bar{D} - \bar{l}^2)\theta = B_0 \bar{w} \quad (27)$$

$$(-\bar{m}^2 - iA_0\bar{m} - \bar{l}^2)\bar{w} = Ra\,\bar{l}^2 F_0 \theta. \quad (28)$$

The above two equations together with the boundary conditions (14) in their dimensionless form define an eigenvalue problem for Ra. For a certain wavenumber $\bar{l}$, a nontrivial solution of the vertical flux and temperature exists only for some values of the Rayleigh number. For a system with given values of T_s, P_s, ΔT, and H, Ra can be defined as a function of $\bar{l}$. The minimum value of Ra for which there is a nontrivial solution is the so-called 'critical Rayleigh number' for the onset of the convective gas flow in a given system.

The condition for the existence of a nontrivial solution of the vertical flux and temperature is that the determinant of the coefficients vanishes, and this leads to a determination of the dimensionless exponent $\bar{D}$:

$$Ra\,\bar{l}^2 B_0 F_0 = (\bar{D} + \bar{l}^2 + \bar{m}^2)(\bar{l}^2 + \bar{m}^2 + iA_0\bar{m}). \quad (29)$$

For stability, $Re(\bar{D}) < 0$, which leads to

$$Ra < \frac{(\bar{l}^2 + \bar{m}^2)^2 + A_0^2\bar{m}^2}{\bar{l}^2 B_0 F_0} = F(\bar{l}^2, \bar{m}^2) \quad (30)$$

for all values of $\bar{l}$ and $\bar{m}$. For any given choice of $\bar{m}$, $F(\bar{l}^2, \bar{m}^2)$ has a minimum at some value of $\bar{l}$. At this minimum

$$\frac{\partial}{\partial \bar{l}^2} F(\bar{l}^2, \bar{m}^2) = 0. \quad (31)$$

The solution of this equation gives

$$\bar{l}^2 = \bar{m}^2 \sqrt{(1 + (A_0/\bar{m})^2)} \quad (32)$$

at which point

$$F = \frac{2\bar{m}^2}{B_0 F_0}(1 + \sqrt{(1 + (A_0/\bar{m})^2)}). \quad (33)$$

The stability limit is most restrictive when $\bar{m} = \pi$, which means the requirement of the boundary conditions at the top and bottom boundaries. The critical Rayleigh number at which instability can first occur is

$$Ra^* = \frac{2\pi^2}{B_0 F_0}(1 + \sqrt{(1 + (A_0/\pi)^2)}). \quad (34)$$

For $A_0 = 0$, $B_0 = 1$ and $F_0 = 1$, which corresponds to the case where the gas is incompressible and there is no vapor pressure in the medium (equivalent to a porous medium saturated with water), we recover Lapwood's [3] critical Rayleigh number of $4\pi^2$.

For a system with $\Delta T = 50$ K, we have $A_0 = 0.8348$, $B_0 = 39.68$, and $F_0 = 2.10$ at the bottom of the layer. The critical Rayleigh number is then 0.25. This compares with $4\pi^2$ in a dry, non-condensible ideal gas (Nield [5]). Physically, the two-order-of-magnitude reduction in critical Rayleigh number reflects the destabilizing effect of latent heat transport in the moist system. The solution is illustrated using streamfunction and isotherm in Fig. 2.

MORE EXACT SOLUTION

The coefficients A_0, B_0, E_0 and F_0 which appear in equations (24) and (25) are functions of z which can be evaluated by equations (18)–(21). We can improve the accuracy of our solution by taking the variation of these quantities into account instead of approximating them by constants.

The dependence of A_0, B_0, and F_0 on temperature is illustrated in Fig. 3. The coefficients A_0 and F_0 vary nearly linearly with the vertical coordinate, but B_0, representing the vapor pressure effect, is highly nonlinear and increases quickly as the vertical coordinate increases (corresponding to an increase in temperature). Because the boundary conditions require that there is no temperature perturbation on the top and bottom surfaces, we can assume a general solution for temperature field in the form of

$$\theta = \sum_j \Theta_j \sin j\pi\zeta. \quad (35)$$

Equations (24) and (25) then can be rewritten as

$$\sum_j (\bar{D} + j^2\pi^2 + \bar{l}^2)\Theta_j \sin j\pi\zeta = -B_0\bar{w} \quad (36)$$

$$\left(\frac{d^2}{d\zeta^2} - (A_0 + E_0)\frac{d}{d\zeta} - \frac{dA_0}{d\zeta} + A_0E_0 - \bar{l}^2\right)\bar{w} = Ra\,\bar{l}^2 F_0 \sum_j \Theta_j \sin j\pi\zeta. \quad (37)$$

Now if we insert the form

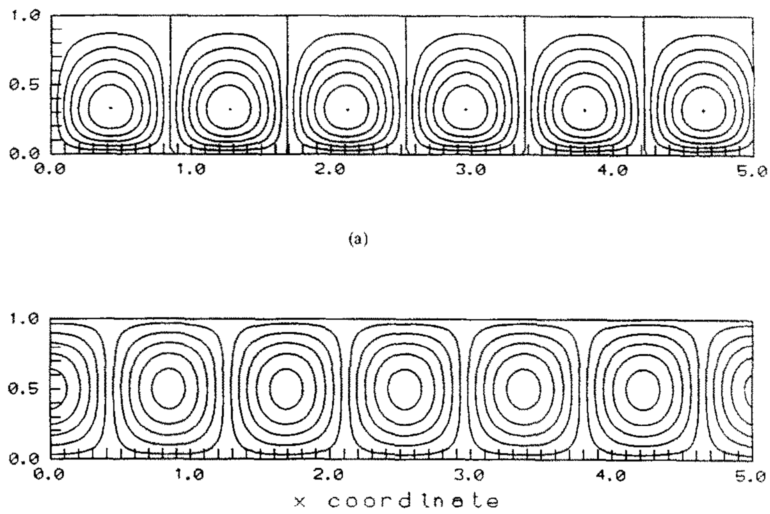
$$\bar{w} = Ra\,\bar{l}^2 \sum_k \Theta_k W_k(\zeta) \quad (38)$$

into equation (37), we have

$$\left(\frac{d^2}{d\zeta^2} - (A_0 + E_0)\frac{d}{d\zeta} - \frac{dA_0}{d\zeta} + A_0E_0 - \bar{l}^2\right)W_k = F_0 \sin k\pi\zeta \quad (39)$$

which is required to satisfy the boundary condition

$$W_k|_{\zeta=0,1} = 0, \quad \bar{w}|_{\zeta=0,1} = 0. \quad (40)$$



(b)

FIG. 2. (a) Streamlines and (b) isotherms at the onset of convection for constant coefficients. The critical Rayleigh number is 0.25, and its critical wavelength is 3.25. The streamlines show two counterrotating convection cells and are plotted for values of the stream function $\phi = \exp(-A\zeta)\sin(\pi\zeta)\sin(\bar{l}x/H)$. The isotherms represent the temperature perturbation solution and are $\theta = \sin(\pi\zeta)\cos(\bar{l}x/H)$. The isotherms reflect only the temperature perturbations due to the gas circulation.

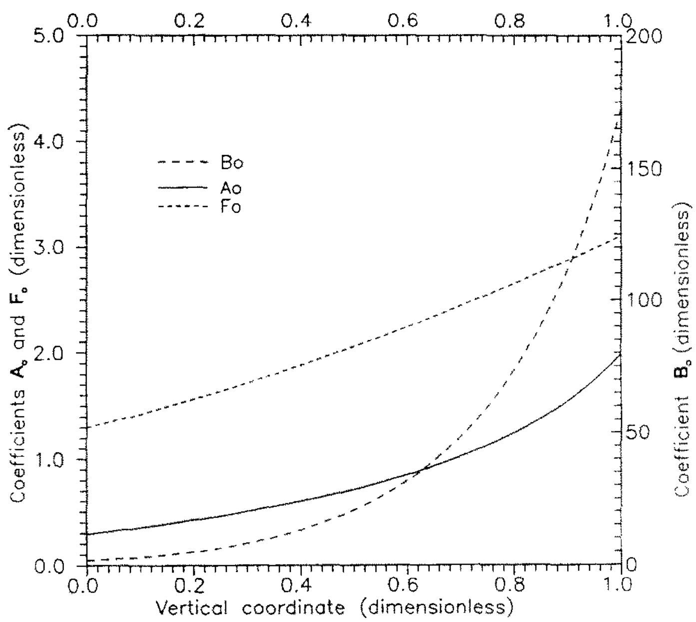


FIG. 3. The coefficients A_0, B_0 and F_0 as functions of temperature (implicitly in the coordinate z).

Equation (39) is a second-order ordinary differential equation with variable coefficients. To solve for the W_k, we employ the **WKB** method, which gives

$$W_k = \frac{e^{(1/2)\int_0^\zeta (A_0+E_0)\,d\zeta}}{f^{1/4}}\left(c_1(\zeta)\,e^{\int_0^\zeta \sqrt{(f)}\,d\zeta} + c_2(\zeta)\,e^{-\int_0^\zeta \sqrt{(f)}\,d\zeta}\right)$$

$$f = \frac{1}{2}\frac{d}{d\zeta}(A_0 - E_0) + \frac{(A_0-E_0)^2}{4} + \bar{l}^2$$

$$c_1(\zeta) = c_1^0 + \frac{1}{2}\int_0^\zeta f^{-1/4}\exp\left(-\int_0^\zeta \sqrt{(f)}\,d\zeta - \frac{1}{2}\int_0^\zeta (A_0+E_0)\,d\zeta\right)F_0 \sin k\pi\zeta\,d\zeta$$

$$c_2(\zeta) = c_2^0 - \frac{1}{2}\int_0^\zeta f^{-1/4}\exp\left(\int_0^\zeta \sqrt{(f)}\,d\zeta\right.$$

$$-\frac{1}{2}\int_0^{\zeta}(A_0+E_0)\,\mathrm{d}\zeta\Bigg)F_0\sin k\pi\zeta\,\mathrm{d}\zeta. \quad (41)$$

The coefficients c_1^0 and c_2^0 can be determined by the boundary conditions (40)

$$c_1^0 = -c_2^0 = \frac{1}{4}\sinh^{-1}\left(\int_0^1 \sqrt{(f)}\,\mathrm{d}\zeta\right)\Bigg\{\Bigg[\int_0^1 f^{-1/4}$$

$$\times\exp\left(\int_0^{\zeta}\sqrt{(f)}\,\mathrm{d}\zeta-\frac{1}{2}\int_0^{\zeta}(A_0+E_0)\,\mathrm{d}\zeta\right)$$

$$\times F_0\sin k\pi\zeta\,\mathrm{d}\zeta\Bigg]e^{-\int_0^1\sqrt{(f)}\,\mathrm{d}\zeta}-\Bigg[\int_0^1 f^{-1/4}$$

$$\times\exp\left(-\int_0^{\zeta}\sqrt{(f)}\,\mathrm{d}\zeta-\frac{1}{2}\int_0^{\zeta}(A_0+E_0)\,\mathrm{d}\zeta\right)$$

$$\times F_0\sin k\pi\zeta\,\mathrm{d}\zeta\Bigg]e^{\int_0^1\sqrt{(f)}\,\mathrm{d}\zeta}\Bigg\}. \quad (42)$$

To solve Θ_k, we substitute equation (38) into equation (36) and obtain

$$\sum_j(\bar{D}+j^2\pi^2+\bar{l}^2)\Theta_j\sin j\pi\zeta = -B_0\,Ra\,\bar{l}^2\sum_k\Theta_k W_k. \quad (43)$$

From the theory of the Fourier sine series transform, we have

$$B_0W_k = \sum_j\left(2\int_0^1 B_0W_k\sin j\pi\zeta\,\mathrm{d}\zeta\right)\sin j\pi\zeta. \quad (44)$$

Substituting the above equation into equation (43) gives

$$\sum_j(\bar{D}+j^2\pi^2+\bar{l}^2)\Theta_j\sin j\pi\zeta+2\,Ra\,\bar{l}^2\sum_k\sum_j\Theta_k$$

$$\times\left(\int_0^1 B_0W_k\sin j\pi\zeta\,\mathrm{d}\zeta\right)\sin j\pi\zeta = 0. \quad (45)$$

Because each term must vanish individually, the above equation can also be written as

$$\sum_{k=0}^{\infty}\Pi_{jk}\Theta_k = 0$$

$$\Pi_{jk} = (\bar{D}+j^2\pi^2+\bar{l}^2)\delta_{jk}+2\,Ra\,\bar{l}^2\int_0^1 B_0W_k\sin j\pi\zeta\,\mathrm{d}\zeta. \quad (46)$$

A nontrivial solution exists when the determinant of the matrix of coefficients Π_{jk} vanishes, i.e.

$$\|\Pi_{jk}\| = 0. \quad (47)$$

A first rank solution of the eigenvalue problem will be given by setting Π_{11} equal to zero and ignoring all the others. This corresponds to the choice of $\sin\pi\zeta$ as a trial function for θ. The corresponding result is

$$\bar{D}+\pi^2+\bar{l}^2+2Ra\bar{l}^2\int_0^1 B_0W_1\sin\pi\zeta\,\mathrm{d}\zeta = 0. \quad (48)$$

The critical Rayleigh number is where $\bar{D} = 0$, or

$$Ra^* = \frac{\pi^2+\bar{l}^2}{2\bar{l}^2\displaystyle\int_0^1 B_0W_1\sin\pi\zeta\,\mathrm{d}\zeta}. \quad (49)$$

Figure 4 shows the critical Rayleigh number as a function of horizontal wave number for different values of temperature difference by using equations (44) and (49). For a system with a specified temperature difference, if the Rayleigh number is above the critical curve, the system will be unstable and form convection cells. Because of the non-linear dependence of vapor pressure on temperature, the higher the temperature difference is, the lower the critical Rayleigh number will be.

When the system's Rayleigh number is equal to the critical Rayleigh number, the system is at the threshold of convection. The neutrally stable solution at this value is depicted in Fig. 5. The streamfunction is solved as $\phi_1 = \exp(-\int_0^{\zeta}A_0\mathrm{d}\zeta)W_1\sin(\bar{l}x/H)$. The isotherms are obtained as $\theta_1 = \sin(\pi\zeta)\cos(\bar{l}x/H)$.

Comparing Fig. 5 with Fig. 2, we notice that the variable coefficient solution has a larger ratio of horizontal wavelength to vertical wavelength than the constant coefficient solution and the location of the stagnation point is higher.

By introducing additional terms in the expansion for θ, we can obtain a more accurate solution for Ra. For the second rank eigenvalue problem, we have

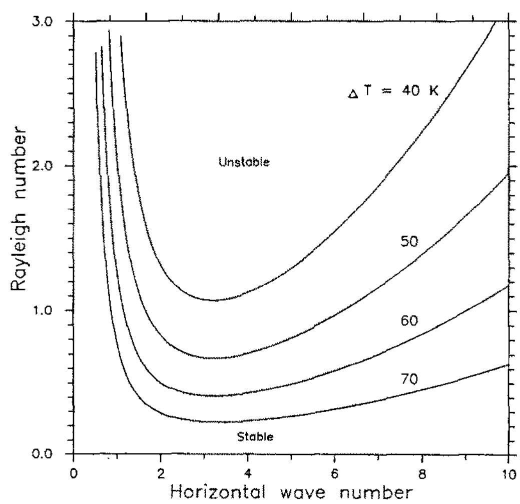
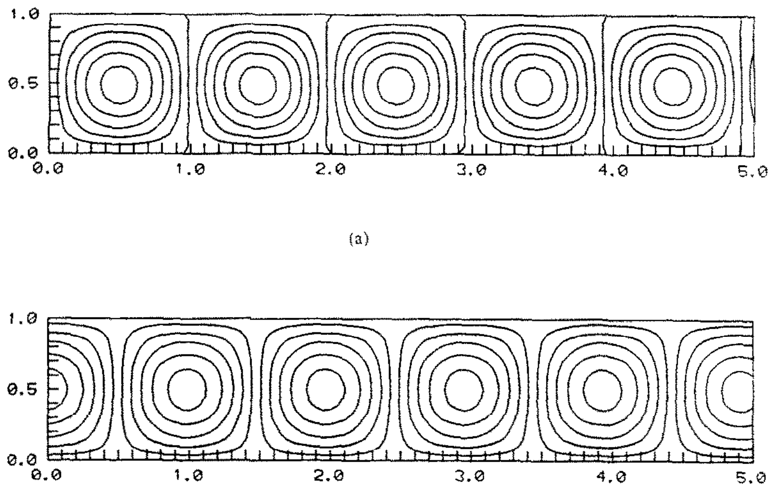


FIG. 4. Boundary between stable and unstable Rayleigh number as a function of horizontal wave number for four values of temperature difference.



(b)

FIG. 5. Streamlines and isotherms for a general eigenvalue problem. The streamfunction is $\phi_1 = \exp(-\int_0^\zeta A_0\,d\zeta)W_1 \sin(\bar{l}x/H)$, and isotherms are $\theta_1 \sin(\pi\zeta)\cos(\bar{l}x/H)$.

$$\begin{vmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{vmatrix} = \begin{vmatrix}\bar{D}+\pi^2+\bar{\Gamma}^2+2\,Ra\bar{\Gamma}^2 I_{11} & 2\,Ra\bar{\Gamma}^2 I_{12}\\ 2\,Ra\bar{\Gamma}^2 I_{21} & \bar{D}+4\pi^2+\bar{\Gamma}^2+2\,Ra\bar{\Gamma}^2 I_{22}\end{vmatrix} = 0$$

$$I_{ij} = \int_0^1 B_0 W_i \sin j\pi\zeta\, d\zeta. \quad (50)$$

The above condition gives the critical Rayleigh numbers corresponding to the first and the second instability modes

$$Ra^*_{1,2} = \frac{-[(\pi^2+\bar{\Gamma}^2)I_{22}+(4\pi^2+\bar{\Gamma}^2)I_{11}] \mp \sqrt{([(4\pi^2+\bar{\Gamma}^2)I_{11}-(\pi^2+\bar{\Gamma}^2)I_{22}]^2+4(\pi^2+\bar{\Gamma}^2)(4\pi^2+\bar{\Gamma}^2)I_{12}I_{21})}}{4\bar{\Gamma}^2(I_{11}I_{22}-I_{12}I_{21})}. \quad (51)$$

Figure 6 shows the temperature field and streamfunction corresponding to the second rank eigenvalue problem.

For the k^{th} rank solution, equation (46) forms a k^{th} rank determinant and can be solved numerically to obtain the system's eigenvalues and the corresponding eigenfunctions.

DISCUSSION

Because it is expected that the higher rank modes will not play as important a role as the lower rank modes, the above calculations with first- and second-order should give a good estimate of the stability of the system. In fact, by comparing the results of the first and the second order approximations, we find that the predictions of both the critical Rayleigh numbers and convection cell patterns are very close to each other. For example, with a 50 K temperature difference and conditions similar to Yucca Mountain (see Table 1), the critical Rayleigh number for the first mode is 0.68 by the first rank, and 0.58 by the second rank. The wavelength ratio (vertical to horizontal) is 1.0 by the first rank, and 1.12 by the second rank. This indicates that the approximations made in the calculations lead to relatively small errors.

As defined in equation (26), a system's Rayleigh number is proportional to the gas permeability of the porous medium. For each temperature difference, there exists a critical Rayleigh number as defined by equation (49). This critical number also depends on other parameters of the system, but less sensitively. Table 2 gives critical Rayleigh numbers for the parameter values given in Table 1. If $\Delta T = 50$ K, the critical permeability is about 2.1×10^{-7} cm^2, and it decreases to about 5.1×10^{-8} cm^2 if $\Delta T = 70$ K. The latter value falls well within the range of gas permeabilities measured at Yucca Mountain [11]. Applications of these results to prediction of heat transfer at Yucca Mountain are discussed by Ross *et al.* [15].

Because the heat source at Yucca Mountain will be transient, it is important to know how quickly convection will develop. To estimate the time constant for the growth of a fluctuation, we treat the case where the Rayleigh number of a system is twice the critical Rayleigh number. From equation (48), we have $\bar{D} = \bar{\Gamma}^2+\bar{m}^2 \simeq 2\pi^2$. Therefore, the rate of growth of the perturbation can be estimated as

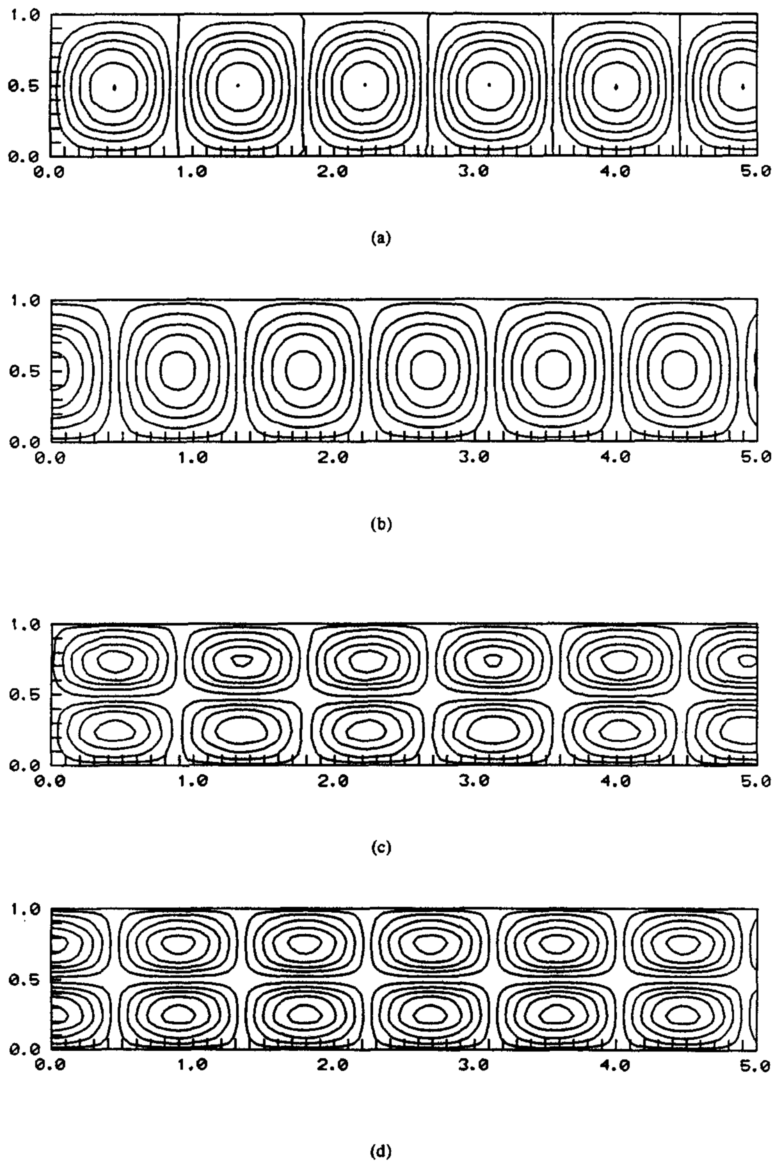


FIG. 6. Streamlines and isotherms for a second rank general eigenvalue problem. (a) The first mode streamfunction is $\phi_1 = \exp(-\int_0^\zeta A_0 \mathrm{d}\zeta) W_1 \sin(\bar{l}_1 x/H)$, and (b) isotherms are $\theta_1 = \sin(\pi\zeta)\cos(\bar{l}_1 x/H)$, (c) the second mode streamfunction is $\phi_2 = \exp(-\int_0^\zeta A_0 \mathrm{d}\zeta) W_2 \sin(\bar{l}_2 x/H)$, and (d) isotherms are $\theta_2 = \sin(\pi\zeta)\cos(l_2 x/H)$.

$$\gamma = \frac{K_t \bar{D}}{c_p^{\text{rock}} \rho_{\text{rock}} (1-n) H^2} \approx \frac{4 \times 10^{-3} \times 2\pi^2}{0.25 \times 3.0 \times (1-0.1)(6 \times 10^4)^2}$$
$$= 4.87 \times 10^{-11}\,\text{s}^{-1} \approx 1/975\,(\text{yr}^{-1}). \quad (52)$$

This indicates that in a system with a Rayleigh number twice the critical value, fluctuations will grow by a factor of e every 975 years.

In mine waste piles, the size and temperature differences are smaller than are foreseen at Yucca Mountain. But the permeability of piles of broken rock can be very large, increasing the Rayleigh number above the critical value. Because the spatial scale is smaller, convection begins much more quickly than in the Yucca Mountain case. Temperature patterns indicating the presence of convection cells have been observed in a field study [18]. Because the heat source is spatially distributed, the mathematical treatment presented here must be extended [19].

CONCLUSIONS

In this study, we use the perturbation method to solve the stability equations analytically for a moist ideal gas heated from below. Dimensionless parameters and coefficients are identified to characterize the stability of the system, and conditions for the onset of convection are obtained. The system must be solved approximately, but comparison of first- and second-

Table 1. Parameters used in the analytic analysis

c_p^{gas}	2.4×10^{-1} cal g^{-1} K^{-1}
c_p^{rock}	2.5×10^{-1} cal g^{-1} K^{-1}
g	9.8×10^{2} cm^{2} s^{-1}
H	6.0×10^{4} cm
H_v	5.39×10^{2} cal g^{-1}
k	$10^{-7} \sim 10^{-6}$ cm^{2}
K_t	4.0×10^{-3} cal K^{-1} cm^{-1} s^{-1}
n	4.0×10^{-1} dimensionless
P_s	8.88×10^{5} g cm^{-1} s^{-2}
R	8.314×10^{7} g cm^{-2} s^{-2} mol^{-1} K^{-1}
T_s	3.0×10^{2} K
ΔT	$40 \sim 70$ K
μ	1.86×10^{-4} g cm^{-1} s^{-1}
ρ_s	1.0×10^{-3} g cm^{-3}
ρ_{rock}	3.0 g cm^{-3}
Ω_a	2.9×10 g mol^{-1}
Ω_v	1.8×10 g mol^{-1}

Table 2. Critical Rayleigh number as a function of the temperature difference

ΔT (K)	Ra^* (1st mode)	$Ra^\dagger$ (2nd mode)
40	0.9499	4.0000
50	0.5782	2.9272
60	0.3397	2.1650
70	0.1792	1.5962

rank solutions show that the approximations yield reasonably accurate solutions.

The formula for the critical Rayleigh number for a system with 100% humidity is more complicated than in a gas without water vapor. The critical Rayleigh number depends on both the wavenumber (as in the simpler case) and four dimensionless coefficients A_0, B_0, E_0 and F_0. Among these dimensionless quantities, B_0, reflecting the change of vapor pressure due to temperature, has the greatest influence on the stability of the system. Increase in vapor pressure tends to destabilize the system. With temperature differences of 40 to 70 K, the critical Rayleigh number is about two orders of magnitude smaller in a moist system than in a dry one. This reflects the greater magnitude of latent heat transport compared to sensible heat transport in high-temperature convection.

The results show that convective instabilities can occur in systems with properties comparable to those occurring in mine-waste piles and at the potential nuclear waste repository site at Yucca Mountain.

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